



Kerma Technology Ltd.

AN002

The maximum power transfer theorem

Introduction

The maximum power transfer theorem shows that when, and only when, the source and load resistance are equal, is the maximum power delivered to the load from that source. This relates to real world signal generators and power supplies where the devices are not perfect and they exhibit some degree of output resistance.

Some time back I was asked by a graduate engineer how was the maximum power transfer theorem proven and this application note came about as a result of my explanation to him. For someone who had done an RF degree at a respectable university, this caused me to question exactly what is taught during university lectures.

The theorem

Figure 1 shows a circuit consisting of a 1V source, that has an output resistance of 1Ω , connected to a resistive load shown here as potentiometer VR1.

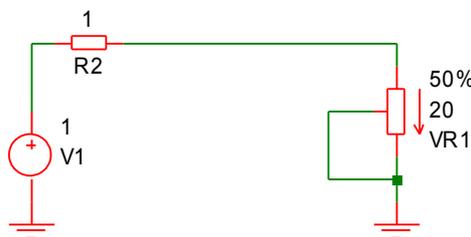


Figure 1: a typical real world voltage source with a fixed output resistance connected to a variable load.

As the potentiometer is varied from zero to its maximum value it is obvious that the current goes from a maximum of 1 Amp, determined by the generator resistance (here, 1Ω), to a value of 47.62mA, or 1v divided by 21Ω . The voltage across the load rises from 0V to a value determined by the potentiometer formed by R2 and VR1. As the power dissipated in the load is the current through it multiplied by the voltage across the load resistance, this rises from zero up to a maximum and then falls back towards zero as the load resistance gets large. The power dissipated will equal zero only when the load is zero or infinite, i.e. a short or open circuit. A plot of the load voltage, current and power dissipated in the load is shown in figure 2 as the load is varied from 0Ω to 20Ω .

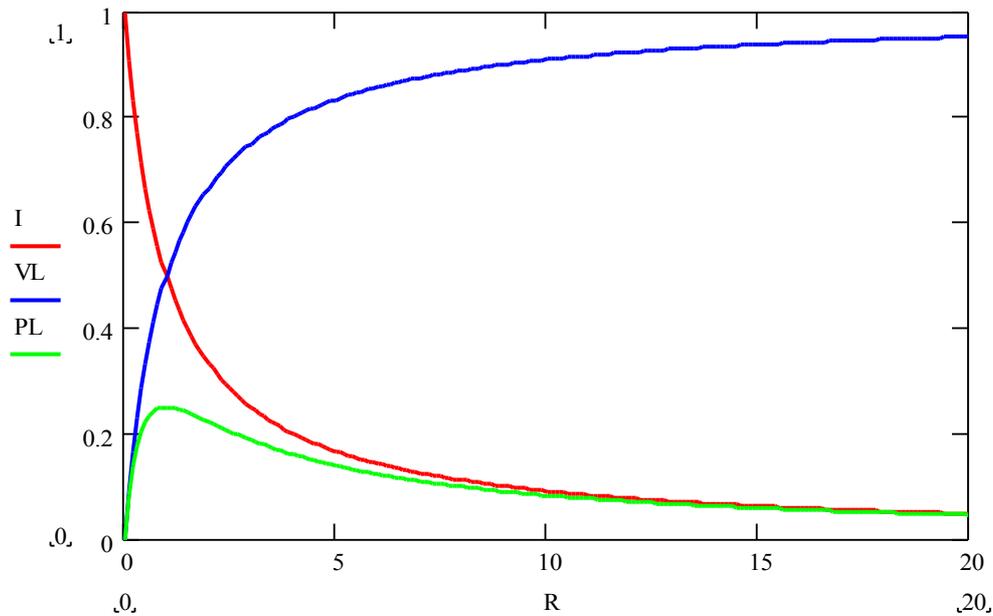


Figure 2: series current, I; load voltage, VL and power dissipated in the load, PL plotted as the load is varied from 0Ω to 20Ω along the x axis.

Note that the power dissipated in the load, the green trace in figure 2, rises to a maximum value and then falls away again. This agrees with the explanation above, there can be no power dissipation in a short or an open circuit. In between these values there will be power delivered to the load.

In order to calculate the point at which the power is maximum, it is necessary to analyse the circuit and then differentiate the equation. Now, don't you wish that you had paid attention back in maths lessons?

The current through the circuit is

$$I = \frac{V}{R_2 + VR_1} \quad [1]$$

The power dissipated in VR₂ is therefore

$$P = I^2 * VR_2$$

Substituting for I in [1]:

$$P = \frac{V^2}{(R_2 + VR_1)^2} * VR_1 \quad [2]$$

Referring back to the graph of figure 2, the region of interest is the peak point of the green trace. This is where the gradient of the slope is zero. By inspection, the slope starts as a positive value and ends as a negative one. So, to find the point where the slope is zero it is necessary to convert equation [2] until it shows gradient. This is done by differentiating the power with respect to the value of VR₁, the load. The point where the gradient is zero must be the peak. The first stage is to differentiate [2] with respect to VR₁.

$$\frac{dP}{dVR1} = \frac{d}{dVR1} \left(\frac{V^2 * VR1}{(R2 + VR1)^2} \right) \quad [3]$$

In this case the result is

$$\frac{dP}{dVR1} = \frac{V^2 * ((R2 + VR1)^2 - VR1 * 2 * (R2 + VR1))}{((R2 + VR1)^2)^2} \quad [4]$$

If you have forgotten how to evaluate a differentiation like that, I've been kind to you, go to the appendix at the end of this document.

The point of interest is when the gradient is zero, therefore the right hand side of [4] can be equated to zero as follows:

$$\frac{V^2 * ((R2 + VR1)^2 - VR1 * 2 * (R2 + VR1))}{[(R2 + VR1)^2]^2} = 0 \quad [5]$$

R2 is a constant and VR1 can only have real, positive, values. Therefore the denominator can be eliminated. Likewise, the entire numerator is multiplied by V². This can be eliminated also. This leaves:

$$(R2 + VR1)^2 - VR1 * 2 * (R2 + VR1) = 0 \quad [6]$$

Simplifying by expanding the terms in brackets and moving the negative term to the right hand side:

$$R2^2 + 2 * R2 * VR1 + VR1^2 = 2 * R2 * VR1 + 2 * VR1^2 \quad [7]$$

Simplifying this equation using normal arithmetic procedures:

$$R2^2 = VR1^2 \quad [8]$$

$$\pm R2 = \pm VR1 \quad [9]$$

Equation [9] has been written with both positive and negative term deliberately, as the square of a negative number gives a positive result also. However, the resistances in a real circuit can only be positive, so the negative answers can be discarded. It is a valid result mathematically to have negative numbers as the answer (or even a negative answer and a positive answer) but such negative resistances cannot exist in this real world application.

So, in conclusion, the point where the gradient is zero, which is also the maximum on the power versus load resistance curve, occurs when R2 is equal to VR1.

Appendix – So, you can't remember how to differentiate?

Like many other engineers, some of the mathematical processes that I used with a degree of fluency when doing O and A-level maths at school, have been forgotten due to having had almost no need to use them in the ensuing decades. Whilst consulting at a firm in Birmingham, I was offered some mathematical books by a colleague who was emigrating and needed to declutter. The lady in question used to help teach mathematics, at evening classes, to students needing to catch up or ones that needed to be given a grounding in the basics. She was offering standard school texts used to teach students up to O- and A-level. Having a house with adequate amounts of junk of my own, I asked only for any book that covered calculus. I received an odd look and was told, “we don't teach calculus up to A-level”. I hope this was just a local aberration and not indicative of the state of the country as a whole. She did add that teaching calculus was standard in Singapore where she was heading to. Enough digression and moaning about education standards.

The equation that needed to be differentiated was [3], repeated here for clarity.

$$\frac{dP}{dVRI} = \frac{d}{dVRI} \left(\frac{V^2 * VRI}{(R2 + VRI)^2} \right) \quad [3]$$

For clarity and to mimic most text books, I will make “VR1” equal to “x”, R2 is a fixed value and becomes constant k.

$$\frac{dP}{dx} = \frac{d}{dx} \left(\frac{V^2 * x}{(x+k)^2} \right) \quad [10]$$

Equation 10 can be simplified further by moving the term V^2 . V is a constant, so V^2 is a constant also. Inspecting equation [10], it is obvious that the term V^2 is simply a scaling factor and can be moved out of the differential term as it does not have an effect on the actual differentiation process. It does have an effect on the final (numerical) answer, but as this treatise is about the differentiation, it has been removed (more accurately, made equal to 1) for clarity. To show this mathematically:

$$\frac{d}{dx} \left(\frac{V^2 * x}{(x+k)^2} \right) = V^2 * \left(\frac{d}{dx} \left(\frac{x}{(x+k)^2} \right) \right)$$

This leaves the simplified equation

$$\frac{dP}{dx} = \frac{d}{dx} \left(\frac{x}{(x+k)^2} \right) \quad [11]$$

First, expand the denominator:

$$\frac{dP}{dx} = \frac{d}{dx} \left(\frac{x}{x^2 + 2xk + k^2} \right) \quad [12]$$

This expression is of the form x divided by a function of x and falls under the quotient rule. Details of this can be found in numerous text books, including the two given in the references section.

Stated simply, the quotient rule is:

If

$$y = \frac{u(x)}{v(x)}$$

Then

$$\frac{dy}{dx} = \frac{\left(v \frac{du}{dx} - u \frac{dv}{dx}\right)}{v^2} \quad [13]$$

Referring back to [12], u is equivalent to the numerator, x, and v is the denominator.

$$u = x$$

$$\frac{du}{dx} = 1$$

and

$$v = x^2 + 2xk + k^2$$

$$\frac{dv}{dx} = 2x + 2k$$

These results can be substituted into equation [13].

$$\frac{dy}{dx} = \frac{\left(\left(x^2 + 2xk + k^2\right)1 - x(2x + 2k)\right)}{\left(x^2 + 2xk + k^2\right)^2} \quad [14]$$

To prove that this is correct, substitute VR1 for x and R2 for k. This equation is then of the same form as [4].

Finally, as all maths teachers used to write, QED.

References

- [1] Weltner, K. et al “*Mathematics for engineers and scientists*” 1986. Stanley Thornes (Publisher) Ltd. ISBN 0-85950-120-5
- [2] Pedoe, J. “*Advanced national certificate mathematics*”. 1955. Vol.1, English Universities Press Ltd.